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Deformation for Positive Characteristic Varieties by Galois theory

Kazuhisa Maehara*

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Abstract

In this article we shall construct a theory for defomation of non necessarily projective fibre spaces of positive characterristic and apply it to an analogy of Iitaka-Viehweg conjecture of positive characteristic([Iita],[Vieh]).

1 Introduction

We construct a deformation theory by making use of axiomatic Galois theory in Seminaire Geometrie Algebrique 1([SGA]) developed by Grothendieck. We obtain the following:

Theorem 1 *Let k be an algebraically closed field of characteristic $p > 0$ and X a projective variety over k.Then the set of projective varieies Y of general type whose m-th pluri-canonical invertible sheaves $\omega_Y^{\otimes m}$ give birational embedding into projective spaces such that there exist dominant separable rational maps from X to Y is finite.*

Theorem 2 *Let k be an algebraically closed field of characteristic $p > 0$. Let $f : X \to S$ be a connected and geometrically irreducible and generically separable morphism between connected and irreducible and generically separable projective schemes over k which we call these varieties. Assume that X is a normal variety and that S is regular. Let $\omega_{X/S}^{[m]}$ denote the double dual of the m-th tensor power of $\omega_{X/S}$ and $\hat{\mathcal{F}}$ the double dual of a torsion free sheaf $\mathcal{F}$. If $\kappa(X_{\bar{\eta}}) \geq 0$ where $\bar{\eta}$ is the generic geometric point of S,then*

$$\max_{m>1}(\kappa(\hat{\det} f_*\omega_{X/S}^{[m]})) \geq \mathrm{var}(X/S)$$

Theorem 3 *Let S be a geometrically irreducible and connected separable scheme over $\mathrm{Spec}\mathbb{Z}$ and X , Y schemes which are geometrically irreducible and connected separable over S and $f : X \to Y$ a geometrically surjective separable morphism over S. Assume that X is isomorphic to a product $X_0 \times_{\mathrm{Spec}\mathbb{Z}} S$, where X_0 is a geometrically irreducible and connected separable scheme over $\mathrm{Spec}\mathbb{Z}$ and that for any geometric point $\bar{s}$ over each point s of S, $\mathrm{Aut}(Y_{\bar{s}})$ is a locally algebraic group over $\bar{s}$. Then there exists a connected separable scheme Y_0 over $\mathrm{Spec}\mathbb{Z}$ such that $Y_{S'}$ is isomorphic to a product $Y_0 \times_{\mathrm{Spec}\mathbb{Z}} S'$ where S' is a scheme surjective onto S.*

2 Preliminary

To apply it to our problems, recall a Grothendieck-Galois theory in expose V le Groupe Fondamental SGA1([SGA]) as preliminary which is axiomatic and has a potentially wide application, though it seems to be not yet enough to apply it even in the area of algebraic geometry. Let **C** be a category and F a covariant functor from **C** to the category **Sets** of sets.

G1 **C** has a terminal object and the fibre product of two objects over a third object in **C**, In other words, there exists a finite limit in **C**.

G2 The finite sum in **C** exists. Hence there exist the initial object $\emptyset_{\mathbf{C}}$ and the quotient of an object of **C** by a finite group of automorphisms. In other words there exist the finite inductive limits in **C**.

* Part-time lecturer, General Education and Research Center, Tokyo Polytechnic University
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G3 Let $u : X \to Y$ be a morphism of $\mathbf{C}$. Then u factorizes to a composition of $X \xrightarrow{u'} Y' \xrightarrow{u''} Y$ with u' a strict epimorphism and u'' a monomorphism, which is an isomorphism on the direct factor of Y.

G4 The functor F is left-exact.

G5 F commutes with finite direct sums, transforms strict epimorphisms to epimorphisms and commutes with fibre products. In other words the functor F is right-exact.

G6 Let $u : X \to Y$ be a morphism in $\mathbf{C}$ such that $F(u)$ is an isomorphism. Then u is an isomorphism.

Then the functor F is strictly pro-representable([GG]). So there exists a projective system

$$P_F = ((P_F)_i)_{i \in I}$$

in $\mathbf{C}$ over I which is ordered and filtered. We have a functorial isomorphism

$$F(X) = \mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}(P_F, X)$$

We call F the fundamental functors, which form a groupoid and P_F the fundamental pro-objects associated to F. We call $\mathrm{Aut}(F)$ the fundamental group which we denote π. It is also the opposite to $\mathrm{Aut}(P_F)$.

Let $\mathbf{C}(\pi)$ be the category of finite sets where π acts continuously to the left. We can consider F as a covariant functor

$$F : \mathbf{C} \to \mathbf{C}(\pi)$$

We can define an inverse functor G

$$G : \mathbf{C}(\pi) \to \mathbf{C}$$

by the formula

$$G(E) = P \times_\pi E$$

F and G are quasi-inverse each other.

Definition 1 *A category which is equivalent to a category $\mathbf{C}(\pi)$ is said to be Galois category, where π is a profinite group.*

The category Pro$-\mathbf{C}(\pi)$ is equivalent to the category $\mathbf{C}'(\pi)$ of the spaces with a topological group π of operators which are compact and totally disconnected. $\mathbf{C}(\pi)$ is a full subcategory of $\mathbf{C}'(\pi)$. π acts to the right on an object X in $\mathbf{C}$ and to the left on $F(X)$. In $\mathbf{C}(\pi)$ a Galois object is isomorphic to a quotient of π by a normal subgroup and a pointed object X has an element a of $F(X)$. Thus a connected pointed object identyfied with an open subgroup of $\mathrm{Aut}(F)$.

Let $\mathbf{C}$, $\mathbf{C}'$ be two Galois categries, $H : \mathbf{C} \to \mathbf{C}'$ a covariant functor, F' a fundamental functor on $\mathbf{C}'$ and $F = F' \circ H$. Then the following conditions are equivalent:

1. H is exact.

2. H is a fundamental functor.

${}^tH$ defines a continuous homomorphism

$${}^tH : \pi_{F'} \to \pi_F$$

Conversely, given two fundamental functors F, F' of $\mathbf{C}$, $\mathbf{C}'$ and a continuous homomorphism $\pi_{F'} \to \pi_F$, it corresponds to a functor of $\mathbf{C}(\pi)$ to $\mathbf{C}(\pi')$ such that it is exact and such that it corresponds to a exact functor $H : \mathbf{C} \to \mathbf{C}'$ such that ${}^tH : \pi_{F'} \to \pi_F$ is u itself.

Proposition 1 *1. Let X be a connected pointed object of $\mathbf{C}$ which is associated to an open subgroup U of π_F. $U \supseteq u(\pi_{F'})$(resp. U contains the closed normal subgroup generated by $u(\pi_{F'})$) iff $H(X)$ admits a pointed section(resp. $H(X)$ is completely decomposed).*

2. *u is trivial iff for any object X of $\mathbf{C}$, $H(X)$ is completely decomposed.*

3. *Let X' be a connected pointed object of $\mathbf{C}'$ which is associated to an open subgroup of $\pi_{F'}$. $U' \supseteq \ker u$ iff there exist a connected pointed object X of $\mathbf{C}$ and a pointed homomorphism of the connectedpointed component X'_0 of $H(X)$ into X'. Hence X is isomorphic to a quotient of the neutral connected component of the inverse image of a pointed object of $\mathbf{C}$ as a pointed object. If u is surjective, X is isomorphic to $H(X)$, where X is a pointed object in $\mathbf{C}$.*

4. *$U' \supseteq \ker u$ iff there exist a connected object X of $\mathbf{C}$ and a morphism of a connected component of $H(X)$ into X'. If u is surjective, X' is isomorphic to an object of the form $H(X)$.*

5. *u is injective iff for any object X' of $\mathbf{C}'$, there exist an object X of $\mathbf{C}$ and a homomorphism of one connected component of $H(X)$ into X'.*

6. *Any connected object of $\mathbf{C}'$ is isomorphic to an object of the form $H(X)$ iff u is isomorphic from $\pi_{F'}$ to a subgroup which is a direct factor of π_F. In fact we however construct a homomorphism $\pi_F \to \pi_{F'}$ which is a right-inverse by the certain exact functor of $\mathbf{C}'$ to $\mathbf{C}$.*

Proposition 2 *The following conditions are equivalent:*

1. *The homomorphism $u : \pi_{F'} \to \pi_F$ is surjective.*

2. *For any connected object X of $\mathbf{C}$, $H(X)$ is connected.*

3. *The functor H is fully faithful.*

Corollary 1 *The following conditions are equivalent:*

1. *u is an isomorphism.*

2. *H is an categorical equivalence.*

3. *The following two conditions are satisfyed:*

   (a) for any connected object X of $\mathbf{C}$, $H(X)$ is connected.

   (b) any object of $\mathbf{C}'$ is isomorphic to an object of the form $H(X)$.

Let S be a locally noetherian connected scheme and

$$a : \mathrm{Spec}(\Omega) \to S$$

a geometric point of S with values in algebraically closed field Ω. Put $\mathbf{C}$ the category of etale coverings of S and for an object X of $\mathbf{C}$ $F(X)$ the set of the geometric points of X over a. Then these $\mathbf{C}$ and the functor F satisfy the axioms (G1)-(G6). Hence it forms a Galois category. Hence the pro-object P_F of $\mathbf{C}$ which represents F is the universal covering of S at the point a. $\pi_F = \mathrm{Aut}(F) = \mathrm{Aut}(P)^o$ is called fundamental group of S at a which is denoted by $\pi_1(S, a)$. If a' is another geometric point of S with values in algebraically closed field Ω', it defines a fundamental functor F'. We denote by $\pi_1(S; a.a')$ the set of homomorphisms $F \to F'$, which are indeed isomorphisms. Now let

$$f : S' \to S$$

a morphism between locally noetherian connected schemes,a' a geometric poit and $a = f(a')$ its imagein S. Then the inverse image functor $f^{\cdot}$ induces a functor of the category $\mathbf{C}(S)$ of the etale coverings of S to the category $\mathbf{C}'(S')$ of the etale coverings of S'. We have an isomorphism of functors

$$F \cong F' \circ f^{\cdot}$$

Hence $f^{\cdot}$ is an exact functor. In particular we have a canonical homomorphism

$$u = \pi_1(f; a'); \pi_1(S', a') \to \pi_1(S, a)$$

which admits to reconstruct inverse image functor $\mathbf{C}(S) \to \mathbf{C}'(S')$.

Proposition 3 *Let S be the spectre of a field k and Ω an algebraically closed extension of k which defines a geometric point a of S. Let $\bar{k}$ be the separable closure in Ω. Then there exists a canonical isomorphism of $\pi_1(S,a)$ onto the topological Galois group of $\bar{k}/k$.*

3 Applications

We propose to give the following definitions for varieties which are not necessarily commutative.

Definition 2 *Let k be a field. A geometrically irreducible and reduced projective schemes over k is called variety.*

A variety V is called of general type if every generically finite cover of V has a finite birational automorphism group.

V is called non-uniruled if every generycally finite cover has a birational automorphism group which is locally algebraic.

V is called uniruled if there exists an infinite sequence of generically finite covers which have birational automorphism groups which are not locally algebraic.

V is called rationally connected if there exist an infinite sequence of generically finite covers the ramification locus of which are dense and whose birational automorphism groups are not locally algebraic.

Theorem 4 *Let k be an algebraically closed field of characteristic $p > 0$ and X a projective variety over k.Then the set of projective varieies Y of general type whose m-th pluri-canonical invertible sheaves $\omega_Y^{\otimes m}$ give birational embedding into projective spaces such that there exist dominant separable rational maps from X to Y is finite.*

Lemma 1 *Let k be an algebraically closed field of characteristic $p > 0$. Let S be the spectre of a function field of a variety T over k and K a function field of S. Let X_0 be a variety over k,and $k(X_0)$ a function field of X_0. Let X be a variety over S such $k(X)$ is isomorphic to the total fractional field of $k(X_0)\otimes_k K$. Let Y be a variety over S. Assume $f : X \to Y$ is a dominant separable rational map over S and assume that the birational automorphism group of Y over S is a locally algebraic group over S. Then there exist a variety Y_0 over k and a variety T' generically finite over T such that the function field $K'(Y_{S'})$ of $Y_{S'}$ is isomorphic to the total fractional field of $k(Y_0)\otimes_k K'$, where the spectre S' of the function field K' of T'.*

Proof 1 *We denote by $\bar{K}$ the separable closure in an algebaically closed extension of K. Let $\bar{Y}$ denote $Y \otimes_K \bar{K}$. We make use of Breen's extension theory([Breen1],[Breen2], [Gir]) We denote by $\mathrm{Aut}(G)$ the automorphism group of the absolute Galois group $G = \mathrm{Gal}(-/\bar{K}(\bar{Y}))$ of $\bar{K}(\bar{Y})$ and by $\mathrm{Out}(G)$ the outer automorpism group of G which is the quotient of $\mathrm{Aut}(G)$ by its inner automorphism. Let P denote the absolute Galois group of K. We have the extension class of P by G in the cohomology*

$$H^1(P,(G \to \mathrm{Aut}(G)))$$

where $(G \to \mathrm{Aut}(G))$ is a crossed module.

$$\begin{tikzcd}
 & H^1(P,Z(G)[1]) \arrow[d] & \\
H^1(P,\mathrm{Aut}(G) \arrow[r] \arrow[dr] & H^1(P,(G \to \mathrm{Aut}(G))) \arrow[r] \arrow[d] & H^2(P,G) \\
 & H^1(P,\mathrm{Out}(G)) &
\end{tikzcd}$$

Since G is a topological group, we define the unique subgroup of $\mathrm{Aut}(G)$ *whose elements* $\phi : G \to G$ *are continuous automorpisms and we have the subgroup of* $\mathrm{Out}(G)$ *corresponding to the subgroup above, which we donote by* $\mathrm{Out}(G)^{cont}$*. Note that*

$$\mathrm{Out}(G)^{cont} \cong \mathrm{Aut}(\mathrm{Spec}\bar{K}(\bar{Y}))$$

In fact $\mathrm{Aut}(\mathrm{Spec}\bar{K}(\bar{Y}))$ *is a subgroup of* $\mathrm{Out}(G)$ *and its maximal continuous subgroup. The absotute Galois group of* $\mathrm{Spec}K(X)$ *has a section over the absolute Galois group P of S and so that E of* $\mathrm{Spec}K(Y)$*. Note that E is an extension of P by G. Then a section of P to E gives an continuous automorphism of G as a conjugate action. It defines an element of* $H^1(P, \mathrm{Aut}(G))$ *which gives an element of* $H^1(P, \mathrm{Out}(G))$ *which is in fact an element of* $H^1(P, \mathrm{Out}(G)^{cont}) = H^1(P, \mathrm{Aut}(\mathrm{Spec}\bar{K}(\bar{Y})))$*. Since P is a profinite group ([RBZL], [Shatz])and* $\mathrm{Aut}(\mathrm{Spec}\bar{K}(\bar{Y}))$ *is a locally algebraic group, there exists a variety T' generically finite over T such that the image of the element of* $H^1(P, \mathrm{Out}(G)^{cont})$ *in* $H^1(P', \mathrm{Out}(G)^{cont})$ *is isomorphic to*

$$P' \to 1 \to \mathrm{Out}(G)^{cont}$$

where P' denotes the absolute Galois group of $k(T')$. Hence the extension element of S' by G in the cohomology $H^1(P', (G \to \mathrm{Aut}(G)))$ *which maps an idenity element of* $H^1(P', \mathrm{Out}(G))$ *comes from* $H^2(P', Z(G))$*. Note that*

$$H^2(P', Z(G)) \to H^1(P', (G \to \mathrm{Aut}(G)))$$

is injective.

The associated inverse-image element of $H^2(P', Z(G))$ *has a section; hence it is a semi-direct product. Since $Z(G)$ is commutative, the semi-direct product is a direct product. Therefore the extension element of* $H^1(P', (G \to \mathrm{Aut}(G)))$ *is trivial.*

Hence there exist a variety Y_0 over k and a variety T' generically finite over T such that the function field $K'(Y)$ of $Y_{S'}$ is isomorphic to the total fractional field of $k(Y_0) \otimes_k K'$, where the spectre S' of the function field K' of T'.

Proof 2 (of Theorem) *Since there exist injections* $H^0(Y, \omega_Y^{\otimes m}) \rightarrowtail H^0(X, \omega_X^{\otimes m})$*, we can birationally embed all of Y by the pluri-canonical mappings into a projective space $\mathbf{P}$ with a fundamental invertible sheaf H. Let $Hilb_{\mathbf{P}}^{p(n)}$ denote the Hilbert scheme of $\mathbf{P}$ with a polynomial $p(n)$.*

Let Y_P denote the connected commponent of closure of the image of Y in $\mathbf{P}$. Hence Y_P are varieties.

We have inclusions for $n \geq 1$

$$H^0(Y_P, H^{\otimes n}) \rightarrowtail H^0(Y, \omega_Y^{\otimes n}) \rightarrowtail H^0(X, \omega_X^{\otimes n})$$

Hence the Hilbert polynomial $P(n)$ has a leading coefficient bounded above. Thus Y_P form a bounded family and so the parameter scheme of Y_P is a noetherian scheme. We shall use noetherian induction and we have the following commutative diagram:

$$\begin{tikzcd} X \times T \arrow[rr] \arrow[dr] & & \{Y_P\} \subseteq \mathbf{P} \times T \arrow[dl] \\ & T & \end{tikzcd}$$

where T is a subscheme of some $Hilb_{\mathbf{P}}$ which parameters Y_P. Apply the lemma above to varieties in the commutative triangle and we have an open subset U of T such that Y_P over any k-point of U is isomorphic to some Y_0.

Therefore the set of projective varieies Y of general type whose m-th pluri-canonical invertible sheaves $\omega_Y^{\otimes m}$ give birational embedding into projective spaces such that there exist dominant separable rational maps from X to Y is finite.

Theorem 5 *Let S be a geometrically irreducible and connected separable scheme over* $\mathrm{Spec}\mathbb{Z}$ *and X , Y schemes which are geometrically irreducible and connected separable over S and $f : X \to Y$ a geometrically surjective separable morphism over S. Assume that X is isomorphic to a product* $X_0 \times_{\mathrm{Spec}\mathbb{Z}} S$*, where X_0 is a geometrically irreducible and connected separable scheme over* $\mathrm{Spec}\mathbb{Z}$ *and that for any geometric point $\bar{s}$ over each point s of S,* $\mathrm{Aut}(Y_{\bar{s}})$ *is a locally algebraic group over $\bar{s}$. Then there exists a connected separable scheme Y_0 over* $\mathrm{Spec}\mathbb{Z}$ *such that $Y_{S'}$ is isomorphic to a product* $Y_0 \times_{\mathrm{Spec}\mathbb{Z}} S'$ *where S' is a scheme surjective onto S.*

Proof 3 *Let s be any point on S, and G_X,(resp.G_Y)) a presheaf of profinite groups over X(resp.Y) such that for any subscheme V of X (resp. Y) and a compatible geometric point a_V, $G_X(V) = \pi_1(V, a_V)$ (resp. $G_Y(V) = \pi_1(V, a_V)$). A scheme X(resp.Y,resp.S) corresponds to a presheaf of profinite groups $\mathcal{G}_X$ (resp.$\mathcal{G}_Y$, resp.$\mathcal{G}_S$). A scheme X_s (resp. Y_s) corresponds to an extension of $\mathcal{G}_{S_s}$ by $\mathcal{G}_{X_{\bar{s}}}$ (resp. $\mathcal{G}_{Y_{\bar{s}}}$) as presheaves of profinite groups. Consider the following commutative triangle:*

$$\begin{tikzcd}
 & H^1(\mathrm{G}_{S_s}, Z(\underline{G_{Y_{\bar{s}}}})[1]) \arrow[d] & \\
H^1(\mathrm{G}_{S_s}, \underline{Aut}(\mathcal{G}_{Y_{\bar{s}}})) \arrow[r] \arrow[dr] & H^1(\mathrm{G}_{S_s}, (\underline{G_{Y_{\bar{s}}}} \to \underline{Aut}(\mathcal{G}_{Y_{\bar{s}}}))) \arrow[r] \arrow[d] & H^2(\mathrm{G}_{S_s}, \underline{G_{Y_{\bar{s}}}}) \\
 & H^1(\mathrm{G}_{S_s}, \underline{Out}(\mathcal{G}_{Y_{\bar{s}}})) &
\end{tikzcd}$$

where $\bar{s}$ is a geometric point over s. By assumption X is a product and so $\mathcal{G}_{Y_s}$ has a section, which defines a conjugate continuous action on $\mathcal{G}_{Y_{\bar{s}}}$. Hence it gives an element of the cohomology $H^1(\mathrm{G}_{S_s}, \underline{Aut}(\mathcal{G}_{Y_{\bar{s}}}))$ and its image is in $H^1(\mathrm{G}_{S_s}, \underline{Out}(\mathcal{G}_{Y_{\bar{s}}}))$.

Note that

$$\underline{Out}(\mathcal{G}_{Y_{\bar{s}}})^{cont} \cong \underline{Aut}(Y_{\bar{s}})$$

An element of $H^1(\mathrm{G}_{S_s}, \underline{Out}(\mathcal{G}_{Y_{\bar{s}}}))$ corresponds to a natural transformation of a cnstant functor to a functor:

$$\mathrm{G}_{S_s} \longrightarrow \underline{Out}(\mathcal{G}_{Y_{\bar{s}}}))$$

Thus it is equivalent to a homomorphism:

$$\mathrm{G}_{S_s} \longrightarrow \Gamma(\underline{Out}(\mathcal{G}_{Y_{\bar{s}}})))$$

Note that

$$\Gamma(\underline{Out}(\mathcal{G}_{Y_{\bar{s}}})^{cont}) \cong \Gamma(\underline{Aut}(Y_{\bar{s}})) = \mathrm{Aut}(Y_{\bar{s}})$$

This automorphism group is locally algebraic by assumption. Hence there exists a scheme S' such that $S' \to S$ is surjective and that for any $s \in S'$ each homomorphism

$$\mathrm{G}_{S_s} \longrightarrow \Gamma(\underline{Out}(\mathcal{G}_{Y_{\bar{s}}})))$$

factors through 1. Thus the extension class in $H^1(\mathrm{G}_{S_s}, (\underline{G_{Y_{\bar{s}}}} \to \underline{Aut}(\mathcal{G}_{Y_{\bar{s}}})))$ comes from $H^1(\mathrm{G}_{S_s}, Z(\underline{G_{Y_{\bar{s}}}}))$, which is in fact a trivial since it has a section. Hence the extension class is also trivial.

Let $\underline{G}_{S_t}$(resp.$\underline{G}_{S_t}$) be a presheaf of profinite groups associated to S_t(resp. $S_{\bar{t}}$ for any point $t \in \mathrm{Spec}\mathbb{Z}$(resp.an geometric point $\bar{t}$ over t) and G_t the absolute Galois group associated to a point t. There exists a short exacte sequence

$$1 \to \underline{G}_{S_{\bar{t}}} \to \underline{G}_{S_t} \to G_t \to 1$$

Hence we have an exact sequence

$$\begin{tikzcd}
(Y_0)_{\bar{t}} \in H^1(\underline{G}_{S'_{\bar{t}}}, (\underline{G}_{Y_{\bar{s}}} \to \underline{Aut}(\mathcal{G}_{Y_{\bar{s}}}))) \\
(Y_0)_t \times_t S_t \in H^1(\underline{G}_{S'_t}, (\underline{G}_{Y_{\bar{s}}} \to \underline{Aut}(\mathcal{G}_{Y_{\bar{s}}}))) \arrow[u] \\
(Y_0)_t \in H^1(G_t, (\underline{G}_{Y_{\bar{s}}} \to \underline{Aut}(\mathcal{G}_{Y_{\bar{s}}}))) \arrow[u]
\end{tikzcd}$$

where in the second row the extension class is trivial since for any point s over t the extension class in $H^1(\mathrm{G}_{S'_{\bar{s}}}, (\underline{G}_{Y_{\bar{s}}} \to \underline{Aut}(\mathcal{G}_{Y_{\bar{s}}})))$ is trivial and the first row the image is trivial and so the class is a product $(Y_0)_{\bar{t}} \times S_{\bar{t}}$ since t is a geometric point and hence there exists an element $(Y_0)_t$ in the third row and therefore we get $(Y_0)_t \times_t S_t$ by pushout. Thus we conclude that $Y_{S'}$ is isomorphic to $Y_0 \times S'$ since Y_t is isomorphic to $(Y_0)_t \times_t S_t$ for any $t \in \mathrm{Spec}\mathbb{Z}$ and Y has a global section of S and so $Y_{S'}$.

4 Analogues

Theorem 6 *Let k be an algebraically closed field of characteristic $p > 0$. Let $f : X \to S$ be a connected and geometrically irreducible and generically separable morphism between irreducible and generically separable projective schemes over k. Assume that X is a normal variety and that S is regular. Let $\omega^{[m]}_{X/S}$ denote the double dual of the m-th tensor power of $\omega_{X/S}$ and $\hat{\mathcal{F}}$ the double dual of a torsion free sheaf $\mathcal{F}$. If $\kappa(X_{\bar{\eta}}) \geq 0$ where $\bar{\eta}$ is the generic geometric point of S,then*

$$\max_{m>1}(\kappa(\hat{\det} f_*\omega^{[m]}_{X/S})) \geq \mathrm{var}(X/S)$$

Corollary 2 *Under the same conditions of the theorem we have*

$$\kappa(\hat{\omega}_{X/S}) \geq \kappa(\hat{\omega}_{X_{\bar{\eta}}}) + \mathrm{var}(X/S)$$

Proof 4 *Assume that $\mathrm{var}(X/S) = \dim S$. $f : X \to S$ factors through a dominant rational map and a projection*

$$\begin{tikzcd}
X \arrow[dr, dotted] \arrow[dd, "f"'] & \\
& P^1 \times \cdots \times P^1 \times S \arrow[dl] \\
S &
\end{tikzcd}$$

Take an integral closure X' of $P^1 \times \cdots \times P^1 \times S$ in the function field $k(X)$ of X. Let Z denote $P^1 \times \cdots \times P^1 \times S$.

$$\begin{tikzcd}
X \arrow[r, dotted] \arrow[dr, dotted] \arrow[dd] & X' \arrow[d] \\
& Z = P^1 \times \cdots \times P^1 \times S \arrow[dl] \\
S &
\end{tikzcd}$$

Let H be a very ample invertible sheaf over $P^1 \times \cdots \times P^1$ and take the pull-back of H over $P^1 \times \cdots \times P^1 \times S$ as the same notation. Construct a cyclic cover Y by $H^{\otimes n} = \mathcal{O}_Z(D)$ where D is a regular divisor and p does not divide n. We have the formula $h_*(\omega^{\otimes m}_{Y/S}) = \oplus_{i=0}^{n-1}\omega^{\otimes m}_{X'/S} \otimes H^{\otimes m(n-1)-i}$ over some open subset of Z. Let f' denote the morphism $X' \to S$, $p : X' \to Z$ and $q : Z \to S$, hence $f' = q \circ p$. The generic geometric fibre Y/S is of general type and so $g^*g_*\omega^{[m]}_{Y/S} \to \omega^{[m]}_{Y/S}$ gives a birational map for some m. Thus using $\mathrm{var}(Y/S) \geq \mathrm{var}(X/S)$ which we shall show, we have

$$\max_{m>1}(\kappa(\hat{\det} g_*\omega^{[m]}_{Y/S})) \geq \mathrm{var}(Y/S) = \dim S$$

Hence there exists a very ample invertible sheaf A such that $A \rightarrowtail \omega^{[m]}_{Y/S}$ over Y for some m and we may assume $A \rightarrowtail \omega^{[m]}_{X'/S} \otimes H^{\ell}$ for some $\ell \geq 0$. We may assume $\ell > 0$. Since Z is reular and $p_*\omega^{[m]}_{X'/S}$ is torsin free, $p_*\omega^{[m]}_{X'/S}$ is locally free except a locus of codimension ≥ 2. We may assume we have $A \rightarrowtail p_*\omega^{[m]}_{X'/S} \otimes H^{\otimes \ell}$. The homomophism

$A \rightarrowtail \hat{p}_*\hat{\omega}^{[m]}_{X'/S} \otimes H^{\otimes \ell}$. gives a homomorphism $A \rightarrowtail p_*\omega^{[m]}_{X'/S}$. outside of the pull-back of the support of A on S(resp. H on $P^! \times \cdots \times P^1$) by the projection $Z \to S$ (resp. $Z \to P^! \times \cdots \times P^1$.) In fact since A is very ample, we have $H^1(\mathcal{F}) = 0$ for any coherent sheaf on an affine variety $S \setminus \mathrm{supp}(A)$ and so we have the homomorphism $A \rightarrowtail q_*\hat{p}_*\hat{\omega}^{[m]}_{X'/S}$ on the affine variety.

Hence we have a homomorphism $A \rightarrowtail p_*\omega^{[m]}_{X'/S}$ on an open subset of Z except an closed subset of codimension ≥ 2. Therefore we have $A \rightarrowtail p_*\omega^{[m]}_{X'/S}$ over the whole Z. Therefore we obtain

$$\max_{m>1}(\kappa(\hat{\det} f'_*\omega^{[m]}_{X'/S})) \geq \mathrm{var}(X'/S) = \dim S$$

It remains to show $\mathrm{var}(Y/S) \geq \mathrm{var}(X'/S)$.

Proposition 4 *If the generic geometric fibre of X'/S is of $\max_{m>0}\{\kappa(\omega^{[m]}_{X'_{\bar\eta}})\} \geq 0$ then $\mathrm{var}(Y/S) \geq \mathrm{var}(X'/S)$.*

Proof 5 *If $\mathrm{var}(Y/S) = \dim S$, it remains nothing to prove. If $\mathrm{var}(Y/S) < \dim S$, then there exist a variety T and a projective variety Y_0 which has a surjective morphism onto T with a geometrically irreducible connected separable projective generic geometric fibre. Since the birational automorphism group of the generic geometric fibre of X'/S is locally algebraic,*

$$\begin{tikzcd}
 & & Y'_{S'} \cong (Y_0)_{S'} \arrow[r] \arrow[dll] \arrow[ddl] \arrow[dr] & X'_{S'} \arrow[ddll] \\
Y \arrow[ddd] & & & Y_0 \arrow[ddl] \\
 & S' \arrow[dl] \arrow[dr] & & \\
S & & T &
\end{tikzcd}$$

For a generic geometric point $\bar t$ of T we have

$$\begin{tikzcd}
(Y_0)_{S'_{\bar t}} \arrow[rr] \arrow[dr] & & X'_{S'_{\bar t}} \arrow[dl] \\
 & S'_{\bar t} \arrow[d] & Y_0 \arrow[d] \\
 & \bar t \arrow[r] & T
\end{tikzcd}$$

Since

$$(Y_0)_{S'_{\bar t}} \cong (Y_0)_{\bar t} \times S'_{\bar t}$$

, apply the lemma above to this case. We have $\mathrm{var}(X'/S) \leq \dim T = \mathrm{var}(Y/S)$.

5 Another minimal model

We refer to the minimal model theory by K.Matsuki([Mats],[Iita]). Here we take a minimal model different from the standard one which we refer to.

Theorem 7 *Let k be the complex number field. Let X be a projective variety of $\kappa(X) \geq 0$ over k. Then there exists a normal variety X' such that a sheaf $\omega_{X'}$ is derived from Weil divisor and nef(resp. abundant).*

Proof 6 *Let X has a dominant rational map onto a product of projective lines $P^1 \times \cdots \times P^1 \times P^1$ of the same dimension of X. Let X' be the integral closure of the structure sheaf of this product in the constant sheaf of the rational function field of X over this product([Baki]).*

$$\begin{array}{c} X' \\ \downarrow \\ P^1 \times \cdots \times P^1 \times P^1 \\ \downarrow \\ P^1 \times \cdots \times P^1 \end{array}$$

Let $P' = P^1 \times \cdots \times P^1 \times P^1$ and $P" = P^1 \times \cdots \times P^1$. The ramification locus of X'/P' is the divisor of the support of $\omega_{X'/P'}$.

Hence denoting the morphism X'/P' by $\pi_{X'/P'}$, we can find an effective divisor $D_{P'}$ on P' such that $\kappa(\omega_{P'}(D_{P'})) \geq 0$ and that

$$(\pi_{X'/P'})_*(\omega_{X'} \otimes (\omega_{P'}(D_{P'}))^{-1})^{\otimes m}$$

is nef(resp. abundant) for sufficiently large m.

Note that $\kappa(\omega_{X'}) \geq 0$ implies $\kappa(\omega_{P'}(D_{P'})) \geq 0$. We shall later show $\omega_{P'}(D_{P'})$ is nef(resp.abundant). Hence $\omega_{X'}$ is nef(resp. abundant). It remain to prove $\omega_{P'}(D_{P'})$ with assumption $\kappa(\omega_{P'}(D_{P'})) \geq 0$ is nef(resp. abundant) by induction.

Since $\omega_{P'}(D_{P'})$ has no vertical component respect to $P'/P"$ which is not a pull-back of a hyperplane of $P"$, we can find $D_{P"}$ such that $\omega_{P'}(D_{P'}) \otimes (\omega_{P"}(D_{P"}))^{-1}$ is an invertible sheaf derived from horizontal hyperplane on P' with respect to $P'/P"$. Thus

$$\pi_{P'/P"_*}(\omega_{P'}(D_{P'}) \otimes (\omega_{P"}(D_{P"}))^{-1})^{\otimes m}$$

is nef(resp. abundant).

From $\kappa(\omega_{P"}(D_{P"})) \geq 0$ and induction assumption, we conclude that $\omega_{P'}(D_{P'})$ is nef(resp. abundant). Hence the induction is complete.

6 Non-Commutative Algebraic Geometry

6.1 preliminary

Recall another Grothendieck's generalization of Galois correspondence mainly in chapter 7(p.294) ([CD]). Let A be a small category and if $\mathbb{A}$ is a presheaf of categories over A, we define the category of representations of $\mathbb{A}$ which we denote by Rep($\mathbb{A}$). Let $\mathbb{G}$ be a presheaf of groups(resp. groupoids) over A. The category Rep($\mathbb{G}$) is the category of presheaves over A with an action of $\mathbb{G}$ on the right. Let $\bigtriangledown\mathbb{A}$ be the fibred category associated to $\mathbb{A}$ and we denote $\bigtriangledown\mathbb{G}$ by $B\mathbb{G}$ which is the classifying category associated to $\mathbb{G}$. We fix a classifying space of $\mathbb{G}$, i.e., a presheaf $B\mathbb{G}$ over A equiped with a universal $\mathbb{G}$-torsor $E\mathbb{G} \to B\mathbb{G}$. If $u : A \to B$ is a smooth functor and if B is a local test category, then A is also a local test category and so $B\mathbb{G}$. By this we obtain a theorem of correspondence of Galois with respect to $\mathbb{G}$.

Let W be a fundamental localizer and $\mathcal{H}_W\hat{A}$ denote the localization of the category $\hat{A}$ by the class $W_{\hat{A}}$ of W-equivalences of presheaves over A. An equivalence of Quillen to the left

$$\mathrm{Rep}(\mathbb{G})/E\mathbb{G} \to \hat{A}/B\mathbb{G} : (X, X \to E\mathbb{G}) \mapsto (\mathbb{G}\backslash X, \mathbb{G}\backslash X \to B\mathbb{G})$$

gives the following fonctor with canonical equivalence of categories $\mathcal{H}_W\mathrm{Rep}(\mathbb{G}) \cong \mathcal{H}_W\mathrm{Rep}(\mathbb{G})/E\mathbb{G}$. Thus we have a functor of descent

$$\mathbf{Desc} : \mathcal{H}_W\mathrm{Rep}(\mathbb{G}) \to \mathcal{H}_W\hat{A}/B\mathbb{G}$$

The functor of descent is an equivalence of categories and admits the functor of monodromy as quasi-inverse

$$\mathbf{Mon} : \mathcal{H}_W\hat{A}/B\mathbb{G} \to \mathcal{H}_W\mathrm{Rep}(\mathbb{G})$$

defined by giving the representation $X \times_{B\mathbb{G}} E\mathbb{G}$ which is a $\mathbb{G}$-torsor.

6.2 Non-Commutative Algebraic Geometry

Let k be an algebraically closed field of charactreistic 0 and let $\mathbf{C}$ be the category of the non-commutative algebraic varieties defined over k([Kon]). For every non-commutative variety X as an object of $\mathbf{C}$ and the Galois group of each Galois extension ([Cohn],[Brz]) of the field associated to the generic point of X, we define a presheaf of Galois groups $\mathbb{G}$ over $\mathbf{C}$. Apply A in the preliminary to $\mathbf{C}$ and we obtain the following lemma and theorems similar argument in the proof of commutative algebraic geometry. Hence we state them without proofs. To change fixing a classifying space correspods to the inner automorpism, which determines an equivalence. Let $u : X \to Y$ be a morphism in the category $\mathbf{C}$. Then we have a homomorphism $G_X \to u^{-1}G_Y$ of presheaves of Galois groups associated to X and Y, respectively. Conversely, This homomorphism $G_X \to u^{-1}G_Y$ up to the equivalence determines a morphism u.

Lemma 2 *Let S be a commutative algebraic variety over k and X, Y non-commutative algebraic projective varieties over S. Assume X is birationall equivalent to a product $X_0 \times S$ where X_0 is a projective variety over k and assume that the generic geometric fibre $Y_{\bar{\eta}}$ of Y/S where $\bar{\eta}$ is an algebraically closed extension point of the generic point η of S has the birational automorphism group which is locally algebraic. If there exists a dominant rational map $X \to Y$ over S, then there exist a variety S' and a generically finite surjective morphism $S' \to S$ such that $Y_{S'}$ is birationally equivalent to a product $Y_0 \times S'$ where Y_0 is a projective variety over k.*

Theorem 8 *Let S be a commutative algebraic variety over k and X, Y non-commutative algebraic projective varieties over S. Assume that the generic geometric fibre $Y_{\bar{\eta}}$ of Y/S where $\bar{\eta}$ is an algebraically closed extension point of the generic point η of S has the birational automorphism group which is locally algebraic. If there exists a dominant rational map $X \to Y$ over S, then*

$$\mathrm{var}(Y/S) \geq \mathrm{var}(X/S)$$

Theorem 9 *Let S be a commutative algebraic variety over k and X, Y non-commutative algebraic projective varieties over S. Assume that X is isomorphic to a product $X_0 \times_k S$ over S where X_0 is a non-commutative projective variety over k. Assume that $Y_{\bar{s}}$ where $\bar{s}$ is any geometric point of S has the automorphism group which is locally algebraic. If there exists a geometrically surjective $X \to Y$ over S, then there exist a variety S' and a surjective morphism $S' \to S$ such that $Y_{S'} \cong Y_0 \times_k S'$ where Y_0 is a non-commutative projective varietyover k.*

Remark 1 *Let K be a differential field with generators of finite trancendent degree over the field of complex numbers and S the set of intermideate differential field with the differential automorphism group which is locally algebraic. Then S is countable.*

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